

Directed polymers and interfaces in random media: Free-energy optimization via confinement in a wandering tube

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We analyze, via Imry-Ma scaling arguments, the strong disorder phases that exist in low dimensions at all temperatures for directed polymers and interfaces in random media. For the uncorrelated Gaussian disorder, we obtain that the optimal strategy for the polymer in dimension $1+d$ with $0 < d < 2$ involves at the same time (i) a confinement in a favorable tube of radius $R_S \sim L^{\nu_S}$ with $\nu_S = 1/(4-d) < 1/2$ (ii) a superdiffusive behavior $R \sim L^\nu$ with $\nu = (3-d)/(4-d) > 1/2$ for the wandering of the best favorable tube available. The corresponding free energy then scales as $F \sim L^\omega$ with $\omega = 2\nu - 1$ and the left tail of the probability distribution involves a stretched exponential of exponent $\eta = (4-d)/2$. These results generalize the well known exact exponents $\nu = 2/3$, $\omega = 1/3$, and $\eta = 3/2$ in $d=1$, where the subleading transverse length $R_S \sim L^{1/3}$ is known as the typical distance between two replicas in the Bethe ansatz wave function. We then extend our approach to correlated disorder in transverse directions with exponent α and/or to manifolds in dimension $D+d=d_t$ with $0 < D < 2$. The strategy of being both confined and superdiffusive is still optimal for decaying correlations ($\alpha < 0$), whereas it is not for growing correlations ($\alpha > 0$). In particular, for an interface of dimension $(d_t - 1)$ in a space of total dimension $5/3 < d_t < 3$ with random-bond disorder, our approach yields the confinement exponent $\nu_S = (d_t - 1)(3 - d_t)/(5d_t - 7)$. Finally, we study the exponents in the presence of an algebraic tail $1/V^{1+\mu}$ in the disorder distribution, and obtain various regimes in the (μ, d) plane.

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I. INTRODUCTION

Directed polymers in random media have attracted a lot of interest for many years, either as interesting disordered models or in relation with stochastic growth models [1]. For polymers in dimension $1+d$ described in the continuum limit by the partition function

$$Z = \int \mathcal{D}\vec{r}(s) e^{-\int_0^L ds (d\vec{r}/ds)^2 - \beta \int_0^L ds V[s, \vec{r}(s)]} \quad (1)$$

with an uncorrelated Gaussian random potential

$$\overline{V(s, \vec{r})V(s', \vec{r}')} = \delta(s - s') \delta^d(\vec{r} - \vec{r}'), \quad (2)$$

the phase diagram is the following [1]: in dimension $d > 2$, there exists a phase transition between a free phase at high temperature [2,3], and a pinned phase at low temperature: this phase transition has been studied numerically in $d=3$ [4], exactly on a Cayley tree [5] and on hierarchical lattice [6]. On the contrary, in dimension $d < 2$, there is no free phase, i.e., any initial disorder drives the polymer into a strong disorder phase. The marginal dimension $d=2$ has been controversial and deserves a special discussion [1]. A strong disorder phase is characterized in particular by two exponents ω and ν for the free energy F and the transverse length scale R :

$$F(L) \sim L^\omega, \quad (3)$$

$$R(L) \sim L^\nu \quad (4)$$

with the expected scaling relation $\omega = 2\nu - 1$ [7]. In $1+1$, these exponents ω and ν are exactly known to be $\omega = 1/3$ and $\nu = 2/3$, because in $d=1$, some “miracles” happen in various

methods: via the mapping towards a damped Burgers equation with random forcing, there exists an exact steady-state distribution that fixes the values of the exponents [8]; within the replica framework, there exists exact Bethe ansatz solutions, that have been studied either in unbounded space [9,10], or in bounded space [11]; there exists an exact combinatorial solution at zero temperature [12], as well as other exact results via the correspondence with stochastic growth models [13] in the same Kardar-Parisi-Zhang universality class. So there are plenty of reasons why the exponents are exactly $1/3$ and $2/3$ in $d=1$. However, various questions are still open or under debate [1], concerning the generalizations of these exponents in various ways, namely: (i) in other transverse dimensions d ; (ii) in the presence of transverse correlations, or time correlations; (iii) for manifolds and interfaces of higher internal dimensions D ; (iv) for various initial disorder distributions, presenting for instance algebraic tails.

In this paper, our aim is to present Imry-Ma scaling arguments that allow to analyze these various generalizations in a unified framework, by a proper identification of the underlying optimal strategy in each case. The paper is organized as follows.

In Sec. II, we recall the two “local” Imry-Ma arguments proposed in Ref. [14] for the directed polymer in $1+d$ dimensions in favorable and unfavorable regions. In Sec. III, we propose a global optimization mechanism between the energy gained by a confinement in a favorable “tube” and the global elastic energy to find the best favorable tube available. This strategy fixes a confinement exponent, a global wandering exponent, as well as a free-energy exponent and the form of the left tail for the free-energy probability distribution. In Sec. IV, we generalize our approach to other correlations in

transverse directions and/or to other internal dimensions: we discuss in particular the cases of directed polymers with correlated transverse disorder, of interfaces with random-bond disorder and of interfaces with random-field disorder. In all these cases, we find exponents in agreement with the replica-scaling analysis by Zhang [15], and we thus discuss the dictionary between our approach via Imry-Ma argument in one disordered sample and the Zhang analysis in replica space. Finally, in the appendices, we discuss two other generalizations for directed polymers in dimension $1+d$: the effects of an algebraic tail $1/V^{1+\mu}$ in the disorder distribution is studied in Appendix A, whereas the case of columnar disorder (i.e., time-independent disorder) is discussed in Appendix B.

II. TWO LOCAL IMRY-MA ARGUMENTS FOR FAVORABLE AND UNFAVORABLE REGIONS

In this section, we recall in details the two Imry-Ma arguments proposed in Ref. [14] which constitute the basis of all our discussion. (In Ref. [14], these arguments were given to interpret the two types of solution found via the disorder-dependent variational method.) As in other contexts, the Imry-Ma argument [16] begins with the evaluation of the typical energy associated to the disorder in a certain volume. Here, for a polymer of length l and transverse length r , the dimensional analysis of the correlator (2) yields the following scaling for the typical random energy

$$\left(\int_0^l ds V[s, \vec{r}(s)] \right)_{typ} \sim \pm u \sqrt{\frac{l}{r^d}}, \quad (5)$$

where u is a random variable of order 1. Regions with $u = u_- > 0$ correspond to globally “unfavorable regions,” whereas regions with $u = -u_+ < 0$ correspond to globally “favorable regions.”

In other contexts, such as random-field Ising models [16], the Imry-Ma argument then consists in comparing the energy cost in creating domain walls with the typical energy gained by taking advantage of the favorable fluctuations of the disorder. Here in the polymer context, the energy coming from the disorder has to be compared with the entropy cost, that can take two different forms [14]: for a swollen polymer $r \gg l^{1/2}$, the entropy cost consists in an elastic term Tr^2/l , whereas for a confined polymer $r \ll l^{1/2}$, the entropic cost consists in a confinement term Tl/r^2 . These two possibilities lead to two different Imry-Ma arguments that can be associated to unfavorable and favorable regions [14] as we now explain.

A. Imry-Ma argument with the elastic term for “unfavorable regions”

The free energy of an unfavorable region of length l_- and transverse length r_- is the sum of the elastic term Tr_-^2/l_- that represents an entropic cost, and the energy cost $u_- \sqrt{l_-}/r_-^d$ from the unfavorable fluctuation of the disorder (5):

$$f_- \sim T \frac{r_-^2}{l_-} + u_- \sqrt{\frac{l_-}{r_-^d}}. \quad (6)$$

The minimization with respect to r_- yields, after dropping numerical prefactors

$$r_- \sim \left(\frac{u_-}{T} \right)^{2\nu_-/3} l_-^{\nu_-} \quad \text{with} \quad \nu_- = \frac{3}{4+d}, \quad (7)$$

in particular, $\nu_-(d=1)=3/5$. The corresponding scaling for the free energy (6) of this unfavorable region reads

$$f_- \sim T^{d/(4+d)} u_-^{4/(4+d)} l_-^{\omega_-} \quad \text{with} \quad \omega_- = 2\nu_- - 1 = \frac{2-d}{4+d}, \quad (8)$$

in particular, $\omega_-(d=1)=1/5$. These exponents ν_- and ω_- actually correspond to the direct dimensional analysis of the initial Hamiltonian, and are usually called “Imry-Ma exponents” or “Flory exponents” in the more general context of interfaces and manifolds in random media [17–23]. In dimension $d=1$, these exponents ($\nu_-=3/5, \omega_-=1/5$) are also the exponents predicted for the full polymer by the replica Gaussian variational ansatz with replica symmetry breaking [22], in contrast with the correct exponents ($\nu=2/3, \omega=1/3$) for the full polymer found by the replica symmetric Bethe ansatz solution [9].

Here, we stress that the above Imry-Ma dimensional analysis should not *a priori* be applied blindly to the full polymer, but only to the unfavorable regions. Our conclusion for the moment being is thus the following: if the polymer has to cross an unfavorable region $u_- < 0$, it will behave as follows when the dimension d varies.

(i) For $0 < d < 2$, the polymer will adopt a wandering exponent $\nu_-=3/(4+d) > 1/2$, and the free energy will have for exponent $\omega_=(2-d)/(4+d) > 0$.

(ii) For the marginal case $d=2$, the wandering exponent reaches the free value $\nu_-(d=2)=1/2$ and the free-energy exponent vanishes $\omega_-(d=2)=0$. A more refined analysis thus becomes necessary.

(iii) For $d > 2$ the above Imry-Ma argument that would yield $\nu_- < 1/2$ breaks down, since for a confined polymer, the elastic free energy r_-^2/l_- has to be replaced by the confinement free energy l_-/r_-^2 . However, in this case, the free energy is minimum in the limit $r_- \rightarrow \infty$, that does not correspond to a confined configuration. So at the level of this scaling analysis, the only consistent possibility for $d > 2$ is that the polymer will keep its free exponent $\nu_-=1/2$ corresponding to a finite elastic free energy, and the disorder potential then corresponds to a subleading term of order $l_-^{(2-d)/2}$, which is what happens in the high-temperature phase.

B. Imry-Ma argument with the confinement term for “favorable regions”

The free energy of a favorable region (l_+, r_+) is the balance between the confinement term Tl_+/r_+^2 , representing the entropy loss due to the confinement, and the energy gain $u_+ \sqrt{l_+}/r_+^d$ from a favorable fluctuation of the disorder (5):

$$f_+ \sim T \frac{l_+}{r_+^2} - u_+ \sqrt{\frac{l_+}{r_+^d}}. \quad (9)$$

The minimization with respect to r_+ yields, after dropping numerical prefactors,

$$r_+ \sim \left(\frac{T}{u_+}\right)^{2\nu_+} l_+^{\nu_+} \quad \text{with} \quad \nu_+ = \frac{1}{4-d}, \quad (10)$$

in particular, $\nu_+(d=1)=1/3$. The corresponding scaling for the total free energy of this favorable region reads

$$f_+ \sim -T^{-d\nu_+} u_+^{4\nu_+} l_+^{\omega_+} \quad \text{with} \quad \omega_+ = 1 - 2\nu_+ = \frac{2-d}{4-d}, \quad (11)$$

in particular, $\omega_+(d=1)=1/3$. In contrast with the exponents (ν_-, ω_-) coming from a direct dimensional analysis of the Hamiltonian, the exponents (ν_+, ω_+) take into account the physical idea that it can be better for the polymer to remain confined in a region to take advantage of favorable fluctuations of the disorder. To our knowledge, these exponents (ν_+, ω_+) have not been considered previously, except in Ref. [14] where they have been introduced.

Our conclusion for the moment being is thus the following: if the polymer has to cross a favorable region $u_+ > 0$, it will behave as follows when the dimension d varies:

(i) For $0 < d < 2$, the polymer will adopt a confinement exponent $\nu_+ = 1/(4-d) < 1/2$, and the free energy will have for exponent $\omega_+ = (2-d)/(4-d) > 0$.

(ii) For the marginal case $d=2$, the confinement exponent reaches the free value $\nu_+(d=2)=1/2$ and the free-energy exponent vanishes $\omega_+(d=2)=0$. A more refined analysis thus becomes necessary.

(iii) For $d > 2$ the above Imry-Ma argument that would yield $\nu_+ > 1/2$ breaks down, since the polymer is not confined anymore. If one replaces the confinement term l_+/r_+^2 by the elastic term r_+^2/l_+ , the total free energy will be minimum for $r_+ \rightarrow 0$ corresponding to a confined configuration. So at the level of this scaling analysis, exactly as in unfavorable regions, the only consistent possibility for $d > 2$ is that the polymer will keep its free exponent $\nu_+ = 1/2$ corresponding to a finite elastic free energy, and the disorder potential then corresponds to a subleading term of order $l^{(2-d)/2}$, which is what happens in the high-temperature phase.

C. Discussion

In this section, we have described via Imry-Ma arguments what typical scalings should be expected from a polymer that is obliged to go through a given favorable region or through a given unfavorable region. In dimension $0 < d < 2$, this analysis yields two sets of nontrivial exponents (ν_-, ω_-) and (ν_+, ω_+) for the two types of regions, whereas for $d > 2$, the only self-consistent exponents in the above Imry-Ma scaling analysis are the exponents of the high-temperature phase. This suggests that the pinned phase existing in dimension $d > 2$ at low temperature is very different in nature from the physics in dimension $d < 2$ and requires a different type of analysis. In the following, we will thus only consider the cases $0 < d < 2$, where the disorder is strong at all scales and

changes the free exponents both in favorable and in unfavorable regions.

III. STRUCTURE OF THE FULL POLYMER OF LENGTH L IN DIMENSION $0 < d < 2$

In this section, we consider the standard situation of a polymer of length L whose origin is fixed and whose end point is free. We discuss what is the best strategy to obtain a minimum free energy, in terms of the favorable regions described in the preceding section.

A. Global optimization on scale L and exponents

The simplest strategy that seems optimal at large scale L is the following: the polymer will try to find a favorable region of length $L_+ \sim L$, of transverse length $R_+ \sim L^{\nu_+}$ and of free energy $F_+ \sim -L^{\omega_+}$. The only degree of freedom available to find this very favorable region is the global orientation $R_G \sim L^\nu$, with respect to the horizontal line, of the tube of radius R_+ starting from the origin forming the favorable region. To find the best favorable region available, the polymer can afford a global elastic cost TR_G^2/L that is at most of the same order of magnitude of the free energy $F_+ \sim -L^{\omega_+}$ of the favorable region it is looking for. The balance between these two terms yields the following global transverse distance:

$$R_G \sim L^\nu \quad \text{with} \quad \nu = \frac{1 + \omega_+}{2} = \frac{3-d}{4-d} \quad (12)$$

and the corresponding free energy

$$F \sim -L^\omega \quad \text{with} \quad \omega = \omega_+ = \frac{2-d}{4-d}. \quad (13)$$

These two exponents are thus the generalizations in dimension $0 < d < 2$ of the well known exact exponents $\nu=2/3$ and $\omega=1/3$ in $d=1$ [1,8]. Moreover, our description also yields the subleading transverse length scale

$$R_+ \sim L^{\nu_+} \quad \text{with} \quad \nu_+ = \frac{1}{4-d} \quad (14)$$

representing the radius of the “tube” of the favorable region, that generalizes the transverse length scale $L^{1/3}$ introduced in Ref. [24] to characterize the size of a “family,” i.e., paths having free-energy differences of order $O(1)$. This subleading transverse length scale $L^{1/3}$ was also interpreted in Ref. [1] as the typical distance between two replicas in the Bethe ansatz replica wave function [9], whereas the scale $L^{2/3}$ represent the displacement of all replicas as a whole.

The physical meaning of the present Imry-Ma scaling analysis is thus the following: the configuration of the polymer is determined by a global optimization mechanism at the largest scale; the polymer chooses the best tube of radius $R_+ \sim L^{\nu_+}$ among all tubes available labeled by the global orientation $\rho = R_L/L^\nu$ defined by the transverse distance R_L of the end point. The number of different tubes available for the choice thus scales as

$$N \sim \frac{R^d}{R_+^d} \sim L^\gamma \quad \text{with} \quad \gamma = d(\nu - \nu_+) = \frac{d(2-d)}{4-d}, \quad (15)$$

in particular, $\gamma(d=1)=1/3$. This number is large enough to find a “good” tube in an arbitrary sample.

B. Tail of the distribution of the free energy

The Imry-Ma analysis for favorable regions can be used to study the tail of the probability distribution of the rescaled free energy. Indeed, the asymptotic behavior of the random variable u_+ in the Imry-Ma argument (5) and (9) is expected to follow to the Gaussian distribution

$$P(u_+) \underset{u_+ \rightarrow \infty}{\simeq} e^{-u_+^2}. \quad (16)$$

The same idea has been already used in the context of the random field XY model [25], and in the context of a disordered heteropolymer, where it was shown to be in full agreement with a disorder dependent real-space renormalization analysis (see the Appendix of Ref. [26]). In the present context, we stress that the Gaussian tail (16) is valid for an initial Gaussian disorder, whereas the presence of an algebraic tail of arbitrary order in the initial disorder will generate a different tail, as discussed in Appendix A.

Here, for a Gaussian initial disorder, the Gaussian tail (16) yields, via the change of variables (11), the following decay for the probability distribution of the rescaled free energy $f_+ = F_+/L_+^{\omega_+}$:

$$\mathcal{P}_+(f_+) \underset{f_+ \rightarrow -\infty}{\propto} T^{d/4} (|f_+|)^{(\eta_+/2)-1} e^{-T^{d/2}|f_+|^{\eta_+}}$$

with

$$\eta_+ = \frac{1}{2\nu_+} = \frac{4-d}{2}. \quad (17)$$

At the level of exponents where we work, since $L_+ \sim L$, the tail of the probability distribution $\mathcal{P}(f)$ for the rescaled free energy $f = F/L^\omega$ of a polymer of length L will thus be given by the same form (17),

$$\mathcal{P}(f) \underset{f \rightarrow -\infty}{\sim} \mathcal{P}_+(f). \quad (18)$$

In particular, the exponent in the exponential in one dimension is $\eta_+(d=1)=3/2$, a value that agrees with the replica-scaling predictions [9,15] and with the numerical simulations [27].

However, to be fully consistent with the approach we propose, we should take into account that the polymer actually chooses the best tube among of large number N (15) of independent tubes. In this interpretation, the random variable u_+ is not just a random variable drawn with a distribution having the Gaussian tail (16), but it is the maximal value u_{max} drawn among N independent variables, i.e., its distribution reads

$$\begin{aligned} \rho_N(u_{max}) &= NP(u_{max}) \left[1 - \int_{u_{max}}^{+\infty} du P(u) \right]^{N-1} \\ &\underset{N \rightarrow \infty}{\sim} N e^{-u_{max}^2} e^{-N \int_{u_{max}}^{+\infty} du e^{-u^2}}. \end{aligned} \quad (19)$$

In particular, u_{max} grows logarithmically in N and thus in L (15),

$$u_{max} \sim \sqrt{\ln N} \sim \sqrt{\ln L}. \quad (20)$$

This would lead (11) to a logarithmic correction to the exponent for the free energy (13), i.e.,

$$F \sim -L^\omega u_{max}^{4\nu_+} \sim -L^\omega (\ln L)^{1/\eta_+}. \quad (21)$$

Is this logarithmic correction a reality or an “artifact” of our interpretation? On one hand, the comparison with the results of most other studies on the subject suggests that this logarithmic correction is spurious. On the other hand, within our approach, it is not clear to us why this logarithmic correction should be disregarded. In particular, if the initial disorder distribution is not Gaussian but presents an algebraic tail of index $(1+\mu)$, it is precisely this mechanism of choosing the best variable u_+ that opens the possibility of obtaining different global exponents even if the variance is finite $\mu > 2$, as discussed in Appendix A. We note moreover that the presence of some logarithmic factors coming from extremal statistics has already been proposed and numerically studied for the directed polymer in $1+1$ [28], as well as in another context involving polymers in random media [29]. In conclusion, even if the exact solution at zero temperature [12] has no logarithmic correction, since we are not aware of exact solutions at nonzero temperature, and since the limit of zero temperature cannot be discussed within our approach (see the discussion below in Sec. III C), it seems that the presence of this logarithmic correction for the free energy at nonzero temperature is a possibility that cannot be completely ruled out, at least to the best of our present knowledge. If there is a proof in the future that there is no logarithmic correction, this would probably mean that the “best” tube is not simply the tube having the maximal rescaled variable u , but the tube having the best structure on smaller scales than the global scale.

C. Remarks on the zero-temperature limit

In this paper, we have considered that the temperature appears only in front of the random potential in the partition function (1) and not in front of the Wiener measure for the Brownian paths. In this case, the elastic term is an entropy coming from the probability $e^{-R^2/L}$ to be at distance R in time L for a Brownian motion. In particular, this elastic term is not present at $T=0$, where the problem on the hypercubic lattice in $(1+d)$ consists in finding the best path among the $(1+d)^L$ possible paths, i.e., on the lattice, a path is either allowed or forbidden, there is no elastic energy at $T=0$. This corresponds to the usual model for numerical simulations on directed polymers at $T=0$. However, many authors are also interested by the models where the elastic term R^2/L is an

energy, i.e., in the partition function (1), the temperature appears not only in front of the random potential but as a global factor in front of the two terms in the exponential (1). In this context, the problem at $T=0$ consists in finding the best path that optimizes the sum of the elastic energy and the random energy.

However, in both cases, within our approach, the limit of zero temperature $T \rightarrow 0$ is very singular, because the entropy due to confinement that played a crucial role at $T \neq 0$ disappears at $T=0$. Let us briefly see what happens as $T \rightarrow 0$ in the Imry-Ma arguments energy/entropy that we have discussed. For unfavorable regions, the prefactor of the transverse scale r_- (7) diverges and the prefactor of the free energy (8) vanishes: indeed, at $T=0$, the free energy (6) only contains the random energy from the disorder (and no entropic elastic term anymore): its minimization corresponds to a transverse scale r_- as big as possible, i.e., on a lattice $r_- \sim L_-$ that leads to an energy $e_- \sim u_- L_-^{(1-d)/2}$. For favorable regions, the prefactor of the confinement scale r_+ (10) vanishes, whereas the prefactor of the free energy (11) diverges: indeed, at $T=0$, the free energy (9) only contains the random energy from the disorder (and no entropic confinement cost anymore): its minimization corresponds to a confinement scale r_+ as small as possible, i.e., on the lattice $r_+ \sim 1$ corresponding to a unique path that leads to an energy of order $e_+ \sim -u_+ L_+^{1/2}$.

In conclusion, our description with Imry-Ma arguments cannot be used to understand the zero-temperature limit, even if there are very direct relations between $T \neq 0$ and $T=0$: in $d=1$, the wandering exponent of the best tube at finite temperature $\nu=2/3$ is the same as wandering exponent of the best path at zero temperature, and the subleading transverse scale $R_+ \sim L^{1/3}$ that represents in our approach the confinement scale at nonzero temperature, had been previously identified in zero-temperature numerical simulations as the typical scale for the first excited states of finite energy above the ground state [24,30].

D. What is the substructure of the polymer at smaller scales ?

From the point of view of the Imry-Ma scaling analysis proposed in this paper, it is clear that the global exponents ω (13) and ν (12) are completely constrained by a global optimization mechanism at the biggest scale. However, once the best global tube has been chosen, it seems natural to expect that the polymer has a “cascade” of optimizations to make on smaller and smaller scales within the large scale constraints. In particular, we expect that the extensive contribution to the free-energy comes from the small scales, since the polymer has to gain a finite contribution at each step on average. However, we also expect that the polymer cannot avoid frustration on all scales, i.e., it will be obliged sometimes to cross unfavorable regions characterized by the exponents ν_- and ω_- (7) and (8). A first interesting question is: what is the scale L_- of the biggest unfavorable region inside the globally favorable tube? A more general question concerns the hierarchical organization on smaller and smaller scales. Indeed, the ultrametric tree structure of local optimal paths [31,30] as well as Monte-Carlo simulations at finite temperature [32] suggest that they are favorable and unfavorable regions on

various scales. In conclusion, it would be very interesting to have a statistical description of the substructure of the global tube, but this goes beyond the present work.

E. Discussion of some consequences

The idea of favorables tubes where the polymer remains confined at finite temperature, with a confinement radius L^{ν_+} much smaller than the wandering scale L^{ν} , gives a more precise meaning to the notion of “states” developed in the droplet scaling theory [33,34]. It is thus interesting to mention briefly some important consequences in dimension $0 < d < 2$ with the values of the global exponents that we have obtained. (It would be of course very interesting to give a more precise characterization of the states at low temperature in dimension $d \geq 2$, but this is left for future work.)

1. Statistics of the effective random potential for the end point

The effective Hamiltonian seen by the free end point $\vec{r} = \vec{r}(L)$ can be decomposed into [33,34]

$$H_{eff} = T \frac{\vec{r}^2}{2L} + L^\omega \Phi \left(\vec{\rho} = \frac{\vec{r}}{L^\nu} \right), \quad (22)$$

where the first term represents the elastic free energy and the second term an effective random potential that has been rescaled with the global exponents, and whose statistics has to be elucidated. The rescaled effective potential $\Phi(\rho)$, has been exactly determined in $d=1$ [13]: it is an “Airy process” [13] that behaves locally as a random walk $\sqrt{\rho}$ as $\rho \rightarrow 0$ and that saturates towards a constant at large distances $\rho \rightarrow \infty$, in agreement with the previous numerical study [35]. More generally, in dimension d , the rescaled effective potential is expected to be independent of L at short distance $\rho \rightarrow 0$ [33], and thus the exponent σ defining the power-law behavior of the effective potential at short distances

$$\Phi(\vec{\rho}) \underset{\rho \rightarrow 0}{\propto} |\vec{\rho}|^\sigma \quad (23)$$

is not a new exponent, but is a function of the two basic exponents [33,34],

$$\sigma = \frac{\omega}{\nu}. \quad (24)$$

With the values obtained before Eqs. (12) and (13), we thus obtain in dimension $0 < d < 2$,

$$\sigma = \frac{2-d}{3-d} \quad (25)$$

that generalizes the random walk behavior $\sigma=1/2$ of the $d=1$ case. The physical meaning of the relation (24) is the following [31,33,34]: optimal configurations whose end points are separated by a distance r typically merge at a distance $l \sim r^{1/\nu}$ and are then identical up to the origin, so that their difference in free energy then scales as $l^\omega \sim r^{\omega/\nu}$. For large distance $r \gg L^\nu$ however, the two paths meet only the origin, and thus experience statistically independent disor-

ders. This is why the random potential saturates for large separation [31,32].

2. Large-scale thermal fluctuations

In the droplet scaling theory [33,34], there are large-scale thermal fluctuations that come from the rare nearly degenerate states. In our description with favorable tubes, if we consider the excitations that involve a length l of the polymer: there exists a large number $N \sim l^\nu$ (15) of other less favorable tubes of energies l^ω with transverse distance $\Delta \sim l^\nu$ with respect to the best tube. As a consequence, with probability T/l^ω , there exists an excited tube with a free-energy difference of order T with respect to the best tube [33]. The power-law distribution $P(\Delta)$ of the end-point transverse scale Δ of thermal excitations computed via dynamical field theory [34] can be understood as follows:

$$P(\Delta)\Delta^{d-1}d\Delta = \frac{T}{l^\omega}\theta(l < L) \quad (26)$$

with the corresponding size $l \sim \Delta^{1/\nu}$, so that

$$P(\Delta) \sim \frac{1}{\Delta^b} w\left(\frac{\Delta}{L}\right) \quad \text{with} \quad b = d + \frac{\omega}{\nu} = d + 2 - \frac{1}{\nu}, \quad (27)$$

where the cutoff w comes from the finite size of the polymer $l \leq L$. With the values obtained before Eqs. (12) and (13), we thus obtain in dimension $0 < d < 2$, the power-law exponent

$$b = \frac{2 + 2d - d^2}{3 - d} \quad (28)$$

that generalizes the exponent $b(d=1)=3/2$.

3. Chaos exponent

The sensitivity to small changes in the disorder distribution [24,36–38] or to small changes in temperature [33] have been interpreted at a scaling level as follows: if the small random perturbation ϵ induces a change of order ϵL^α for the free energy of the previous state, there is an instability if $\alpha > \omega$ for lengths larger than $L_c(\epsilon) \sim \epsilon^{-c}$ with the so called chaos exponent $c = 1/(\alpha - \omega)$. For a random-bond perturbation [24,36,37], and for a temperature change [31], the exponent of the perturbation is in both cases $\alpha = 1/2$. For the temperature change, this comes from the behavior of entropy fluctuations $\Delta S \sim L^{1/2}$, that was conjectured in Ref. [33] and numerically checked [33,39]. With the values obtained before (13), the chaos exponent thus reads in dimension $0 < d < 2$:

$$c = \frac{1}{1/2 - \omega} = \frac{2(4 - d)}{d} \quad (29)$$

that generalizes the well-known exponent $c(d=1)=6$. Finally, let us mention that more general perturbations with other exponents α have also been studied in Refs. [37,40], and that the scaling form for the free-energy decorrelation has been recently computed on a Berker lattice [41].

IV. GENERALIZATION TO OTHER DIMENSIONS AND/OR OTHER TRANSVERSE CORRELATIONS

In this section, we generalize our approach to the case of a D -dimensional manifold embedded in a space of total dimension $D + d = d_t$ in the solid-on-solid approximation: the manifold is described by a d -dimensional vector field $r(x)$, where x is the D -dimensional vector of internal coordinates. We consider the standard Hamiltonian

$$H = \int d^D x \sum_{\mu=1}^D \left(\frac{\partial r}{\partial x_\mu} \right)^2 + \int d^D x V[x, r(x)], \quad (30)$$

where $V(x, r)$ is a Gaussian random potential with correlator

$$\overline{V(x, r)V(x', r')} = \delta^D(x - x') C(|r - r'|), \quad (31)$$

where the asymptotic behavior of the correlation in transverse directions is governed by some exponent α ,

$$C(r) \underset{r \rightarrow \infty}{\propto} r^\alpha. \quad (32)$$

For instance, the case of local disorder characterized by δ correlations also in transverse directions corresponds via scaling to the case $\alpha = -d$, whereas a random-field disorder corresponds to $\alpha = 1$.

The dimensional analysis of the elastic term $L^{D-2}R^2$ shows that the free behavior in the absence of disorder is characterized by the exponent

$$\nu_{free}(D) = \frac{2 - D}{2} \quad (33)$$

that generalizes the random walk exponent $\nu_{free}(D=1) = 1/2$ of the polymer case. In the following, we will only consider the cases $0 < D < 2$, where $\nu_{free} > 0$.

A. Local Imry-Ma argument with the elastic term

The generalization of the free energy (6) reads

$$f_- \sim T \frac{r_-^2}{l_-^{2-D}} + u_- l_-^{D/2} r_-^{\alpha/2}. \quad (34)$$

For $\alpha u_- < 0$, the minimization of the free energy with respect to r_- yields

$$r_- \sim u_-^{2/(4-\alpha)} l_-^{\nu_-} \quad \text{with} \quad \nu_- = \frac{4 - D}{4 - \alpha} \quad (35)$$

and the corresponding scaling for the free energy reads

$$\begin{aligned} f_- &\sim u_-^{4/(4-\alpha)} l_-^{\omega_-} \quad \text{with} \quad \omega_- = 2\nu_- - (2 - D) \\ &= \frac{2D + (2 - D)\alpha}{4 - \alpha}. \end{aligned} \quad (36)$$

This solution is consistent as long as $\nu_- > \nu_{free}$, i.e., since $0 < D < 2$,

$$-\frac{2D}{2 - D} < \alpha < 4. \quad (37)$$

The condition $\alpha u_- < 0$ shows that the physical interpretation is very different according to the sign of α . For all $\alpha < 0$, as in the special case of uncorrelated disorder $\alpha = -d$, the solution obtained corresponds to $u_- > 0$, i.e., to an unfavorable region, where the global free energy is positive $f_- > 0$. On the contrary, for $\alpha > 0$, the solution corresponds to $u_- < 0$, i.e., to a favorable region where the global free energy is negative $f_- < 0$.

B. Local Imry-Ma argument with the confinement term

The generalization of the Imry-Ma argument (9) with the confinement entropy reads

$$f_+ \sim T \left(\frac{l_+}{r_+^{2/(2-D)}} \right)^D - u_+ l_+^{D/2} r_+^{\alpha/2}. \quad (38)$$

(A detailed analysis of the confinement entropy can be found in Ref. [42].) For $\alpha u_+ < 0$, the minimization with respect to r_+ yields

$$r_+ \sim u_+^{-(2/D)} \nu_+ l_+^{\nu_+} \quad \text{with} \quad \nu_+ = \frac{D(2-D)}{4D + \alpha(2-D)}. \quad (39)$$

The corresponding scaling for the total free energy reads

$$f_+ \sim u_+^{4D/[4D+(2-D)\alpha]} l_+^{\omega_+}$$

with

$$\omega_+ = D - \frac{2D}{2-D} \nu_+ = \frac{D[2D + (2-D)\alpha]}{4D + (2-D)\alpha}. \quad (40)$$

This solution is consistent as long as $0 < \nu_+ < \nu_{free}$, i.e.,

$$-\frac{2D}{2-D} < \alpha. \quad (41)$$

The condition $\alpha u_+ < 0$ shows that the physical interpretation is again very different according to the sign of α . For $\alpha < 0$, as in the special case of uncorrelated disorder $\alpha = -d$, the solution obtained corresponds to $u_+ > 0$, i.e., to a favorable region, where the global free energy is negative $f_+ < 0$. On the contrary, for $\alpha > 0$, the solution corresponds to $u_+ < 0$, i.e., to an unfavorable region where the global free energy is positive $f_+ > 0$.

C. Global optimization for $\alpha < 0$

For $\alpha < 0$, as in the polymer case with uncorrelated disorder, the optimal strategy will be to find a favorable region $L_+ \sim L$ with confinement $R_+ \sim L^{\nu_+}$. To find the best region available, it will be worth to afford a global elastic cost of $L^{D-2} R_G^2$ of the same order of magnitude of the energy gain $E_+ \sim L^{\omega_+}$ of the favorable region. The balance between these two terms yields that the global wandering will be

$$R_G \sim L^\nu \quad \text{with} \quad \nu = \frac{\omega_+ + 2 - D}{2} = \frac{D(4-D) + (2-D)\alpha}{4D + (2-D)\alpha}. \quad (42)$$

The global free energy gain will be

$$F \sim L^\omega \quad \text{with} \quad \omega = \omega_+ = \frac{D[2D + (2-D)\alpha]}{4D + (2-D)\alpha}. \quad (43)$$

The subleading transverse length scale characterizing the confinement reads

$$R_+ \sim L^{\nu_+} \quad \text{with} \quad \nu_+ = \frac{D(2-D)}{4D + \alpha(2-D)}. \quad (44)$$

Finally, the use of Eq. (16) for the random variable u_+ yields, via the change of variables (40) to the following decay for the probability distribution of the free energy $F \sim F_+$:

$$\mathcal{P}(F) \underset{F \rightarrow -\infty}{\simeq} \frac{1}{|F|} \left(\frac{|F|}{L^\omega} \right)^{\eta_+/2} e^{-(|F|/L^\omega)^{\eta_+}}$$

with

$$\eta_+ = \frac{4D + (2-D)\alpha}{2D}. \quad (45)$$

D. Global optimization for $\alpha > 0$

For $\alpha > 0$, the previous picture completely changes, because the favorable regions are not the confined solutions (+), but the swollen solutions (-). As a consequence, the global optimization coincide with a solution (-): the global exponents for the transverse direction and the free energy are thus directly given by

$$\nu = \nu_- = \frac{4-D}{4-\alpha}, \quad (46)$$

$$\omega = \omega_- = \frac{2D + (2-D)\alpha}{4-\alpha}. \quad (47)$$

Finally, the use of Eq. (16) for the random variable u_- yields, via the change of variables (36) to the following decay for the probability distribution of the free energy $E \sim E_-$,

$$\mathcal{P}(E) \underset{E \rightarrow -\infty}{\simeq} \frac{1}{|E|} \left(\frac{|E|}{L^\omega} \right)^{\eta_-/2} e^{-(|E|/L^\omega)^{\eta_-}} \quad \text{with} \quad \eta_- = \frac{4-\alpha}{2}. \quad (48)$$

We now briefly describe our results with their domain of validity for the interesting cases of a polymer $D=1$ or of interfaces $d=1$, before we compare with the results of other methods.

E. Results for the polymer with decaying correlations with exponent $-2 < \alpha < 0$

For a polymer $D=1$ with correlations described by the exponent α (32) with $-2 < \alpha < 0$, the results are the same as for the polymer with uncorrelated disorder with the replacement $d \rightarrow -\alpha$, i.e., there is confinement with exponent

$$\nu_+ = \frac{1}{4+\alpha} \quad (49)$$

with the corresponding exponents for the free energy and the wandering

$$\omega = 1 - 2\nu_+ = \frac{2 + \alpha}{4 + \alpha}, \quad (50)$$

$$\nu = \frac{3 + \alpha}{4 + \alpha} \quad (51)$$

with the tail exponent $\eta_+ = 2 + \alpha/2$. The domain of validity in α is limited by $\alpha \rightarrow -2$, where the confinement exponent reaches the free random walk exponent $\nu_+ \rightarrow 1/2$, and by $\alpha \rightarrow 0$ where the optimization mechanism changes.

F. Results for the polymer with long-range correlations with exponent $0 < \alpha < 4$

For a polymer $D=1$ with correlations described by the exponent α (32) with $0 < \alpha < 4$, there is no confinement, and the exponents are directly given by the exponents $(-)$ coming from the direct dimensional analysis of the Hamiltonian

$$\nu = \frac{3}{4 - \alpha}, \quad (52)$$

$$\omega = 2\nu - 1 = \frac{2 + \alpha}{4 - \alpha}, \quad (53)$$

and the tail exponent $\eta = (4 - \alpha)/2$. The comparison with Eq. (51) shows that the exponents are continuous at $\alpha=0$ with the values $\nu(\alpha=0)=3/4$ and $\omega(\alpha=0)=1/2$. Special interesting cases discussed in Ref. [1] are $\alpha=1$, and $\alpha=2$. In particular for the latter case $\alpha=2$ that corresponds effectively to the Larkin model, there exists an exact solution in terms of replicas yielding the exact free-energy distribution [43]: our approach is in agreement with the exact results for the global exponents $\nu=3/2$, $\omega=2$ and for the exponent $\eta=1$ of the tail of the free-energy distribution [43].

G. Results for interfaces with random-bond disorder

For the special cases of interfaces of internal dimension $0 < D < 2$ in a space of total dimension $D+1=d_t$, the random-bond disorder corresponds to uncorrelated disorder in the $d=1$ transverse direction, i.e., the effective exponent is $\alpha = -d = -1 < 0$. The constraint (41) leads to the domain of validity $2/3 < D < 2$, i.e., in terms of the total dimension d_t of space

$$\frac{5}{3} < d_t < 3. \quad (54)$$

The confinement exponent reads

$$\nu_+ = \frac{(d_t - 1)(3 - d_t)}{5d_t - 7}. \quad (55)$$

Note that it vanishes in the limit $d_t \rightarrow 3$, because the free exponent ν_{free} also vanishes in this limit of a two-dimensional surface $D \rightarrow 2$ and is replaced by logarithms, the physical meaning being that the confinement is not a big constraint anymore [42]. The corresponding exponents for the free energy and wandering read

$$\omega = \frac{(d_t - 1)(3d_t - 5)}{5d_t - 7}, \quad (56)$$

$$\nu = \frac{7d_t - 8 - d_t^2}{5d_t - 7} \quad (57)$$

with the tail exponent $\eta_+ = (5d_t - 7)/[2(d_t - 1)]$.

H. Results for interfaces with random-field disorder

For the special cases of interfaces of dimension $0 < D < 2$ in a space of total dimension $D+1=d_t$, the random-field disorder that corresponds to the exponent is $\alpha = +1 > 0$. The constraint (37) does not modify the original constraints $0 < D < 2$, i.e., in terms of the total dimension d_t of space

$$1 < d_t < 3. \quad (58)$$

In this case, there is no confinement, and the exponents are given by the exponents $(-)$, i.e., they reduce to the well-known exponents coming from the direct dimensional analysis of the Hamiltonian

$$\nu = \frac{5 - d_t}{3}, \quad (59)$$

$$\omega = \frac{1 + d_t}{3} \quad (60)$$

with the left tail exponent $\eta = 3/2$, that was also found in Ref. [23].

I. Agreement with Zhang replica scaling analysis

The exponents presented in this section are in agreement with Zhang replica-scaling analysis [15]. It is thus useful to describe briefly the “dictionary” between the two methods. Whereas the traditional approach with replicas consists in considering the limit $n \rightarrow 0$ with the possibility of replica symmetry breaking, the replica-scaling analysis proposed by Zhang [15] consists in analyzing the replicated problem within a symmetric treatment of the n replicas with n large [for instance $n(n-1)$ is replaced by n^2]. The leading orders in (L, n) of the moments are then interpreted as coming from the tail of the probability distribution of the free energy. In Zhang analysis, the case $\alpha < 0$ corresponds to an attraction between replicas and leads to a bound state, whereas the case $\alpha > 0$ corresponds to a repulsion between replicas with no bound state. So the presence of a bound state for replicas corresponds in our approach to the presence of a confinement tube. In conclusion, for those who prefer to think in real space than in replica space, our approach provides an equivalent self-contained description in real space; and for those who prefer to think in replica space, we hope that the translation in real space is also interesting, and can perhaps be useful in other disordered models.

J. Comparison with other methods

Within the field theoretical framework, the correctness of the direct dimensional Imry-Ma analysis for the random-field

disorder $\alpha=1$ was supported by a functional RG analysis [18], because the large distance behavior of the correlations are not renormalized, whereas for random-bond disorder $\alpha=-1$, there are nontrivial renormalization of the exponents [18]. So there should be a critical value α_c below which the direct dimensional analysis does not give the correct exponents. It has been argued in various RG methods [19–21,44] that (i) below the critical value α_c , the wandering exponent ν sticks to the random-bond value ν_{RB} and would thus be “superuniversal” (ii) the critical value α_c is strictly negative. However, since the argument by Fisher [18] that the long-range correlations of the disorder are not renormalized, naturally stops to apply at $\alpha=0$, it seems to us that from this point of view, the simplest scenario is that the true critical value is actually $\alpha_c=0$, in agreement with the prediction of the replica-scaling analysis [15] and with our analysis with confinement. Moreover, we believe that our description involving a confinement mechanism suggests that a more appropriate field theoretical description in the case of transverse decaying correlations $\alpha<0$ should perhaps include the presence of two different important transverse length scales, and should perhaps involve instantons calculations or other nonperturbative tools to describe the confinement.

Finally, we should mention that the various alternative predictions [15,44,45] for the exponents in the domain $\alpha<0$ have been tested via numerical simulations on kinetic roughening with various conclusions [46,47]. We believe that our approach, that explains the meaning of Zhang replica-scaling exponents in physical space strongly suggests that these exponents are indeed the correct ones.

V. CONCLUSION

In this paper, we have analyzed via Imry-Ma energy/entropy arguments the strong disorder phases that exist in low dimensions at all nonzero temperatures for directed polymers and interfaces in random media. Within the field of disordered systems, the originality of random manifolds is that they have some freedom to “choose” the disorder variables they see in a given sample, in contrast with spin systems for instance that cannot avoid any disorder variables. Our main result is that they can use this freedom to follow two qualitatively different strategies: (i) for disorder with decaying transverse correlations $\alpha<0$, the optimal strategy consists in being confined in a wandering tube, i.e., there are two different transverse scales that are important, namely, the wandering scale and the confinement scale; (ii) for disorder with long-range transverse correlations $\alpha>0$, the optimal strategy is to be swollen, i.e., there is only one important transverse scale, and the exponents are given by a simple dimensional analysis of the original Hamiltonian. For the general case of a manifold of internal dimension D in a space of total dimension $D+d$ in the presence of transverse correlations of exponent α , our results for exponents agree with Zhang replica-scaling analysis [15]. Our method thus describes the same physics, but in real space instead of being in replica space, and this allows to make the link with the scaling droplet approach: the states at finite temperatures of the scaling droplet theory [33,34] are interpreted in our approach

as Imry-Ma favorable tubes, where the polymer remains confined, and it is this confinement in an Imry-Ma favorable tube that translates into a bound state for replicas in the replica-scaling analysis [15].

In conclusion, we hope that our approach via simple scaling arguments in real space gives a clear insight into the strong disorder phases in low dimensions, and a complementary point of view with respect to the other methods.

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APPENDIX A: INITIAL DISORDER DISTRIBUTIONS WITH ALGEBRAIC TAILS

For the lattice uncorrelated models, where each link has an energy $V(i,x)$ drawn with some symmetric law $P(V)=P(-V)$, the question of the universality with respect to the form of the initial probability distribution $P(V)$ has attracted a lot of interest [1], because it appeared that the presence of long tails in the microscopic disorder

$$P(V) \simeq \frac{1}{|V|^{1+\mu}} \quad (A1)$$

was able to change the exponents [48–50], as well as the morphology of the associated ultrametric tree structure of the locally optimal paths [51], even if the variance was finite $\mu>2$.

1. The strategy of finding the best site energy at zero temperature

For the disorder (A1), the heuristic argument [48,50] that has been proposed to explain these dependences in μ consists in the balance between the maximal site energy $V_{max}=(LR^d)^{1/\mu}$ drawn in the volume (LR^d) and the global elastic energy R^2/L , that yields the exponent

$$\nu_{(V_{max})}^{elastic}(\mu) = \frac{\mu+1}{2\mu-d}. \quad (A2)$$

This exponent corresponds to the strategy of finding the maximal site energy and is usually considered as a lower bound [50] for the true exponent $\nu(\mu)$. For instance in $d=1$, this exponent is bigger than the value $\nu_{Gauss}=2/3$ for $\mu\geq 5$, which means that, at least for $\mu\geq 5$, the true exponent $\nu(\mu)$ is not the usual one $\nu_{Gauss}=2/3$. There has been a large number of numerical studies on the true exponent $\nu(\mu)$, either on the directed polymer problem or in corresponding growth models: it is usually believed [1] that the exponent (A2) is not the exact one [48,50] and in particular that the associated critical value $\mu_c(d=1)=5$ in $d=1$ is too low [52] and should be replaced by at least $\mu_c(d=1)\sim 7$, even if it has been also argued that these numerical conclusions were due to cross-over effects and that the exponent (A2) was exact [53].

From our point of view, it is not very clear why the exponent (A2) should describe the exponents at zero tempera-

ture of the lattice models that are measured in the numerical simulations. Indeed, for lattice models at zero temperature, there is no elastic energy, since a path is either allowed or forbidden. It is thus useful to reconsider in this context the strategy of finding the maximal site energy V_{max} among a number of order L^{1+d} of independent sites that can be reached for a path of length L . The maximal value is thus of order $V_{max} \sim L^{(1+d)/\mu}$, and since the distribution of its position is uniform, it reads in terms of the longitudinal coordinate l and the transverse distance r in continuum notations

$$\rho_L(l, r) \sim \frac{r^{d-1} \theta(r < l < L)}{L^{d+1}}. \quad (\text{A3})$$

As a consequence, the probability distribution of the transverse distance r of the best site V_{max} takes the scaling form

$$\rho_L(r) = \int_0^L dl \rho_L(l, r) \sim \frac{1}{L} \left(\frac{r}{L}\right)^{d-1} \left(1 - \frac{r}{L}\right), \quad (\text{A4})$$

i.e., the exponent corresponding to the strategy of finding the best site energy is

$$\nu_{(V_{max})}^{(\text{lattice})}(\mu, d) = 1 \quad (\text{A5})$$

and any deviation from this extremal value $\nu=1$ on the lattice has to come from some cooperative effect of a large number of subleading best sites. In the following, we will not discuss the zero temperature anymore, but the finite temperature case, and we will try to describe the cooperative effects by reconsidering our previous Imry-Ma arguments to see what changes are necessary in the presence of algebraic tails. So we first need to consider the random energy that can be gained in a tube.

2. Distribution of the energy U of a path of length l in a tube of radius r

Let us consider the random energy

$$U = \sum_{i=1}^l V[i, x(i)] \quad (\text{A6})$$

of a path of length l in a tube of radius r in dimensions $(1+d)$. It is clear that its moments diverge for $k > \mu$ as the moments of the initial law (A1). The tail of its probability distribution will thus present the same power-law decay

$$P(U) \underset{|U| \rightarrow \infty}{\simeq} \frac{c(l, r)}{|U|^{1+\mu}}, \quad (\text{A7})$$

where the scaling of the prefactor $c(l, r)$ with respect to (l, r) can be estimated as follows. Let us introduce temporarily a large cutoff A in the original distribution (A1) to regularize the diverging moments $k > \mu$:

$$\langle V^k \rangle_A \underset{A \rightarrow \infty}{\simeq} A^{k-\mu}. \quad (\text{A8})$$

The largest divergence in the cutoff A of the moment

$$\langle U^k \rangle_A = \sum_{i_1} \dots \sum_{i_k} \langle V[i_1, x(i_1)] \dots V[i_k, x(i_k)] \rangle_A \quad (\text{A9})$$

is then given by the term of order $\langle V^k \rangle_A$, corresponding to coincident indices $i_1 = i_2 = \dots = i_k$ and coinciding transversal positions $x_{i_1} = \dots = x_{i_k}$. If one assumes a uniform confinement in transverse directions within a tube of radius r , we thus obtain

$$\langle U^k \rangle_A \underset{A \rightarrow \infty}{\simeq} A^{k-\mu} \frac{l}{r^{d(k-1)}}, \quad (\text{A10})$$

i.e., the prefactor $c(l, r)$ of the tail (A7) scales as

$$c(l, r) \sim \frac{l}{r^{d(\mu-1)}}. \quad (\text{A11})$$

On the other hand, the largest contribution in l of the moments (A9) with even k (odd moments vanish by symmetry) comes from the term corresponding to $(k/2)$ pairs of coinciding indices, leading to

$$\langle U^k \rangle_A \underset{l \rightarrow \infty}{\simeq} \left(\langle V^2 \rangle \frac{l}{r^d} \right)^{k/2}. \quad (\text{A12})$$

The natural rescaling appropriate for this term corresponds as it should to the rescaled variable u introduced before for the Gaussian case (5)

$$U = u \sqrt{\frac{l}{r^d}}. \quad (\text{A13})$$

However, the rescaled variable u will now present the following algebraic tail (A7) and (A11)

$$P(u) \underset{|u| \rightarrow \infty}{\simeq} \frac{(lr^d)^{1-(\mu/2)}}{|u|^{1+\mu}}. \quad (\text{A14})$$

The presence of a nontrivial radius r thus generalizes the special case $r \sim 1$ corresponding to the classical problem of the sum U of l independent variables:

(i) For $\mu > 2$, the weight $(lr^d)^{1-(\mu/2)}$ of the tail of the rescaled variable $u = U/\sqrt{l}$ vanishes in the limit $l \rightarrow \infty$, i.e., there is a generalized central limit theorem describing the limit law of the rescaled variable u as soon as the variance is finite. However, since the polymer will try to find the best tube available, the presence of the algebraic tail can induce a change in global exponents as we will see.

(ii) For $\mu < 2$, the weight of the tail in Eq. (A14) is a now positive power of l : the variable u is not appropriate anymore, and the new appropriate rescaled variable v is then defined by Eqs. (A7) and (A11)

$$U = vL^{1/\mu} R^{d(1-\mu)/\mu}, \quad (\text{A15})$$

so that the limit probability distribution of v presents the tail

$$Q(v) \underset{|v| \rightarrow \infty}{\simeq} \frac{1}{|v|^{1+\mu}}. \quad (\text{A16})$$

Since the exponent of the transverse scale R in Eq. (A15) changes of sign at $\mu=1$, we will obtain a qualitative change at $\mu=1$.

3. Local Imry-Ma arguments

In this section, we discuss the “local” Imry-Ma arguments in the different cases $\mu > 2$, $1 < \mu < 2$ and finally $0 < \mu < 1$.

a. Local Imry-Ma arguments for $\mu > 2$

For $\mu > 2$, the typical energy gained in a tube is still defined in terms of the rescaled variable u (A13) and (A14), and thus the exponents (ω_+, ν_+) (10) and (11) for typical favorable regions and (ω_-, ν_-) (7) and (8) for typical unfavorable regions are unchanged.

b. Local Imry-Ma arguments for $1 < \mu < 2$

For $\mu < 2$, the new appropriate rescaled variable is v (A15), and thus the typical energy associated to a tube is different from the Gaussian case. The Imry-Ma arguments for favorable and unfavorable regions will thus yield different typical exponents.

(i) *Analysis of the exponents in typical favorable regions.* The free energy of a typical region (l_+, r_+) with the confinement term (9) then becomes

$$f_+ \sim T \frac{l_+}{r_+^2} - v_+ l_+^{1/\mu} r_+^{d[(1/\mu)-1]}. \quad (\text{A17})$$

For a favorable region $v_+ > 0$, the minimization of f_+ with respect to r_+ yields

$$r_+ \sim v_+^{-\mu/[(2-d)\mu+d]} l_+^{\nu_+(\mu)} \quad \text{with} \quad \nu_+(\mu) = \frac{\mu-1}{(2-d)\mu+d}. \quad (\text{A18})$$

The corresponding free-energy reads

$$f_+ \sim -v_+^{2\mu/[(2-d)\mu+d]} l_+^{\omega_+(\mu)} \quad \text{with} \quad \omega_+(\mu) = \frac{2+d-d\mu}{(2-d)\mu+d}. \quad (\text{A19})$$

For $0 < d < 2$, this analysis is valid for $1 < \mu < 2$: the confinement exponent varies between $\nu_+(\mu \rightarrow 1) \rightarrow 0$ and its usual value $\nu_+(\mu \rightarrow 2) \rightarrow 1/(4-d)$, and the free-energy exponent varies between $\omega_+(\mu \rightarrow 1) \rightarrow 1$ and its usual value $\omega_+(\mu \rightarrow 2) \rightarrow (2-d)/(4-d)$. In particular, in dimension $d=1$, the exponents read

$$\nu_+(\mu, d=1) = \frac{\mu-1}{\mu+1}, \quad (\text{A20})$$

$$\omega_+(\mu, d=1) = \frac{3-\mu}{\mu+1}. \quad (\text{A21})$$

For $d \geq 2$, these exponents are expected to be valid as long as the confinement exponent $\nu_+(\mu)$ is positive and smaller than the free value $1/2$, i.e., for $\mu < 1+2/d$. For instance, in transverse dimension $d=2$, the validity domain is $1 < \mu < 2$, whereas in $d=3$ it is $1 < \mu < 5/3$. Equivalently, for fixed μ in the interval $1 < \mu < 2$, the exponents are valid for $d < 2/(\mu-1)$.

(ii) *Analysis of the exponents in typical unfavorable re-*

gions. The free energy of a typical region (l_-, r_-) with the elastic term (6) then becomes

$$f_- \sim T \frac{r_-^2}{l_-} + v_- l_-^{1/\mu} r_-^{d[(1/\mu)-1]}. \quad (\text{A22})$$

For an unfavorable region $v_- > 0$, the minimization of f_- with respect to r_- yields

$$r_- \sim v_-^{\mu/[(2+d)\mu-d]} l_-^{\nu_-(\mu)} \quad \text{with} \quad \nu_-(\mu) = \frac{\mu+1}{(2+d)\mu-d}. \quad (\text{A23})$$

The corresponding free energy reads

$$f_- \sim v_-^{2\mu/[(2+d)\mu-d]} l_-^{\omega_-(\mu)} \quad \text{with} \quad \omega_-(\mu) = \frac{2+d-d\mu}{(2+d)\mu-d}. \quad (\text{A24})$$

For $0 < d < 2$, this analysis is valid for $1 < \mu < 2$: the roughness exponent varies between $\nu_-(\mu \rightarrow 1) \rightarrow 1$ and its usual value $\nu_-(\mu \rightarrow 2) \rightarrow 3/(4+d)$, and the free-energy exponent varies between $\omega_-(\mu \rightarrow 1) \rightarrow 1$ and its usual value $\omega_-(\mu \rightarrow 2) \rightarrow (2-d)/(4+d)$. In particular, in dimension $d=1$, the exponents read

$$\nu_-(\mu, d=1) = \frac{\mu+1}{3\mu-1}, \quad (\text{A25})$$

$$\omega_-(\mu, d=1) = \frac{3-\mu}{3\mu-1}. \quad (\text{A26})$$

c. Local Imry-Ma argument for $0 < \mu < 1$

For $0 < \mu < 1$, the exponent of the transverse scale R in Eq. (A15) is positive. As a consequence, exactly as in the case of long-range transverse correlation, the favorable regions do not correspond to confined solutions, but to swollen solutions. The free-energy of a region (l_-, r_-) with the elastic term (6) has now to be replaced by

$$f_- \sim T \frac{r_-^2}{l_-} - v_- l_-^{1/\mu} r_-^{d[(1/\mu)-1]}. \quad (\text{A27})$$

For a favorable region $v_- < 0$, the minimization with respect to r_- yields

$$r_- \sim v_-^{\mu/[(2+d)\mu-d]} l_-^{\nu_-(\mu)} \quad \text{with} \quad \nu_-(\mu) = \frac{\mu+1}{(2+d)\mu-d} \quad (\text{A28})$$

and the corresponding free-energy reads

$$f_- \sim -v_-^{2\mu/[(2+d)\mu-d]} l_-^{\omega_-(\mu)} \quad \text{with} \quad \omega_-(\mu) = \frac{2+d-d\mu}{(2+d)\mu-d}. \quad (\text{A29})$$

In particular, in the limit $\mu \rightarrow 1$, the free-energy exponent $\omega_-(\mu \rightarrow 1^-) = 1$ is in continuity with the $\omega_+(\mu \rightarrow 1^+) = 1$, Eq. (A19).

4. Global optimization for the full polymer

In the presence of long tails, the global optimization for the full polymer consists in looking for the confinement scale $R_S \sim L^{\nu_S(\mu)}$ and the global wandering scale $R_G \sim L^{\nu(\mu)}$ that minimize the global free energy containing three contributions

$$F_L(R_G, R_S) = T \frac{R_G^2}{L} + T \frac{L}{R_S^2} - U_{max} \left(L, R_S; N = \frac{R_G^d}{R_S^d} \right). \quad (\text{A30})$$

The first term R_G^2/L represents the global elastic cost for the diffusive tube. The second term L/R_S^2 represents the confinement entropy cost. The third term U_{max} represents the best energy among N independent variables U , where U represents the random energy associated to a tube (L, R_S) . Using the form (A7) and (A11) of the tail of its probability distribution, we thus obtain the following scaling:

$$U_{max} \left(L, R_S; N = \frac{R_G^d}{R_S^d} \right) \sim [c(L, R_S)N]^{1/\mu} \sim R_S^{-d} (LR_G^d)^{1/\mu}. \quad (\text{A31})$$

This result shows why the introduction of two transverse scales (R_S, R_G) is actually always better to gain energy than the swollen solution that involves only one scale R_G , whose free energy reads

$$F_L^{swollen}(R_G) = T \frac{R_G^2}{L} - U_{typ}(L, R_G), \quad (\text{A32})$$

where the typical value scales as $U_{typ}(L, R_G) \sim R_G^{-d} (LR_G^d)^{1/\mu}$ for $\mu < 2$ (A15). The comparison with Eq. (A31) is thus immediate: both have the same factor $(LR_G^d)^{1/\mu}$ presenting the scale of the best sites in the volume (LR_G^d) , but the factor of the density in the transverse directions of the tube is much better in a confined tube R_S^{-d} than for the swollen solution R_G^{-d} .

The optimization of Eq. (A30) with respect to (R_S, R_G) yields the following exponents:

$$\nu(\mu) = \frac{1 + \mu(1-d)}{(2-d)\mu - d}, \quad (\text{A33})$$

$$\nu_S(\mu) = \frac{\mu - (d+1)}{(2-d)\mu - d}. \quad (\text{A34})$$

These exponents are valid in transverse dimension $0 < d < 2$ in the domain

$$\mu_s(d) = d+1 < \mu < \mu_c(d), \quad (\text{A35})$$

where the inferior value $\mu_s(d) = d+1$ corresponds to the point where the confinement exponent vanishes $\nu_S(d+1) = 0$, whereas the superior value corresponds the critical value

$$\mu_c(d) = \frac{2(1+d\nu)}{(1+d\nu_+)} = 2 + d - \frac{d^2}{2}, \quad (\text{A36})$$

where the Gaussian exponents are recovered. This critical value varies between $\mu_c(d \rightarrow 0) \rightarrow 2$ and $\mu_c(d \rightarrow 2) \rightarrow 2$, and is maximal for $d=1$ with the value $\mu_c(d=1) = 5/2$. For instance, in dimension $d=1$, the exponents (A34),

$$\nu(\mu, d=1) = \frac{1}{\mu-1}, \quad (\text{A37})$$

$$\nu_S(\mu, d=1) = \frac{\mu-2}{\mu-1} \quad (\text{A38})$$

are thus expected to be valid for $2 \leq \mu \leq \mu_c(d=1) = 5/2$.

Let us now discuss what happens for $\mu < \mu_s(d) = d+1$. Below this value, the confinement exponent would become negative, which is unphysical, and thus the confinement exponent will stick to its minimal value

$$\nu_S[\mu < \mu_s(d) = d+1] = 0 \quad (\text{A39})$$

corresponding to an extreme confinement. In this regime, the confinement entropy of order L is subleading with respect to the elastic entropy and the disorder energy in Eq. (A30). The balance between these two terms then coincides with the estimation (A2) coming from the single best site strategy

$$\nu \left(\frac{d}{2} < \mu < \mu_s(d) = d+1 \right) = \nu_{(V_{max})}^{elastic}(\mu) = \frac{\mu+1}{2\mu-d}. \quad (\text{A40})$$

Note that this wandering exponent is then always greater than one. As a consequence, on a lattice, the wandering exponent will stick to its maximal value $\nu=1$,

$$\nu^{lattice}[\mu \leq \mu_s(d) = d+1] = 1. \quad (\text{A41})$$

5. Discussion

In this appendix, we have shown how the presence of algebraic tails in the initial distribution could be taken into account in our approach to yield a change in the global exponents even in the cases where the variance is finite $\mu > 2$. However, even if our results are on this point qualitatively correct, the status of the quantitative results is not clear. Indeed, in dimension $d=1$, we have obtain as μ varies for the wandering exponent at nonzero temperature

$$\nu(d=1, 2 < \mu < 5/2) = \frac{1}{\mu-1}, \quad (\text{A42})$$

$$\nu(d=1, 1/2 < \mu < 2) = \frac{\mu+1}{2\mu-1} \geq 1, \quad (\text{A43})$$

i.e.,

$$\nu^{lattice}(\mu \leq 2) = 1,$$

where the change that takes place at $\mu=2$ with the corresponding value $\nu(\mu=2)=1$, between a phase with a positive confinement exponent $\nu_s(\mu > 2, d=1) > 0$ and a phase with a

vanishing confinement exponent $\nu_s(\mu < 2, d=1)=0$. The results (A43) for $\mu > 2$ are far from the exponents measured at $T=0$ in numerical simulations: in particular, the critical value we have obtained $\mu_c(d=1)=5/2$ is very far from the lattice numerical estimation $\mu_c(d=1, T=0) \sim 7$ [52,1]. Is there a problem in this case between the zero-temperature best path and our approach at nonzero temperature based on the confinement entropy? Or is our Imry-Ma approach too simple to describe correctly the algebraic tails? For instance, it may be that the smaller scales have to be taken into account, in contrast with the Gaussian case where the optimization on the biggest scale fixes the exponents.

APPENDIX B: CASE OF COLUMNAR DISORDER

Following the suggestion of an anonymous referee, we now briefly discuss the confinement properties of a directed polymer at nonzero temperature in the presence of columnar disorder, i.e., when the random potential $V[s, \vec{r}(s)]$ becomes a function of space $V(\vec{r})$ only (see Refs. [1,54] and references therein, for the physical motivations and the results of various approaches). The case of columnar disorder is known to

be governed by the extremal statistics of the initial distribution [54], and thus to simplify the discussion, we consider the specific case of binary disorder $V(\vec{r}) = \pm v_0$. The probability $P_+(R)$ to have a tube of radius R containing only attractive lines decays exponentially as in the standard Lifshitz argument

$$P_+(R) \sim e^{-R^d}. \quad (\text{B1})$$

The best favorable tube of radius R_S among $N \sim R_G^d/R_S^d$ independent tubes has for radius $R_S \sim (\ln N)^{1/d}$. This best tube allows to reach the best total energy available $-v_0L$, whereas the confinement entropic cost and the elastic cost are, respectively, TL/R_S^2 and TR_G^2/L . The best solution is thus the following at leading order: (i) the confinement radius of the best favorable tube grows logarithmically $R_S \sim (\ln L)^{1/d}$; (ii) the transverse distance between the origin and the best tube grows almost linearly $R_G \sim [L/(\ln L)^{1/d}]$; (iii) the difference between the free-energy and its extensive part decays as $\Delta F \sim [L/(\ln L)^{2/d}]$. The result for the free-energy is in agreement with Ref. [54], whereas the exponent for the logarithmic factor in the total transverse displacement R_G is different from Ref. [54].

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